IOPscience iopscience.iop.org

Home Search Collections Journals About Contact us My IOPscience

The discrete cubic and chiral cubic spin systems in various dimensions and their large-N limits

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

1985 J. Phys. A: Math. Gen. 18 653

(http://iopscience.iop.org/0305-4470/18/4/014)

View the table of contents for this issue, or go to the journal homepage for more

Download details:
IP Address: 129.252.86.83
The article was downloaded on 31/05/2010 at 09:27

Please note that terms and conditions apply.

J. Phys. A: Math. Gen. 18 (1985) 653-660. Printed in Great Britain

The discrete cubic and chiral cubic spin systems in various dimensions and their large-N limits

R Badke, P Reinicke and V Rittenberg
Physikalisches Institut, Universität Bonn, Nussallee 12, D-5300, Bonn 1, West Germany

Received 17 July 1984, in final form 20 August 1984

Abstract. The phase transitions of the cubic and chiral cubic models are studied using mean-field and Monte Carlo combined with finite-size scaling methods. The critical exponent of the specific heat for the $N=3$ cubic model in $d=3$ dimensions is in agreement with the experimental data. The large-N behaviour of both systems suggests that in the large-N limit the mean field is exact.

1. Introduction

Two models for spin systems have been of special interest for workers in the field of particle physics; these are the σ-models and chiral models.

The O(N)-models are a prototype for the σ-models and are defined by the Hamiltonian

$$-H=\sum_{\langle i,j\rangle}(k_1\boldsymbol{S}_i\boldsymbol{S}_j+k_2(\boldsymbol{S}_i\boldsymbol{S}_j)^2) \tag{1.1}$$

where the sum is over the nearest neighbours and $\boldsymbol{S}_i$ is an N-component vector of unit length. We have stopped at the first two terms in the powers of $(\boldsymbol{S}_i\boldsymbol{S}_j)$.

Chiral models are defined taking a group G with elements g and defining the Hamiltonian:

$$-H=\sum_{\langle ij\rangle}\sum_{\lambda}a_\lambda\chi_\lambda(g_ig_j^{-1}) \qquad (g\in \mathrm{G}) \tag{1.2}$$

where χ_λ is the character function corresponding to the irreducible representation λ of G. The Hamiltonian (1.2) has G$\otimes$G as symmetry. In this paper we concentrate on the case where the group G is O(N).

The partition function is defined as

$$Z=\sum \mathrm{e}^{-H/kT} \tag{1.3}$$

and we take ferromagnetic interactions.

The original reason for our study was to check if it is possible to approximate the continuous manifolds which are the N-dimensional sphere (for equation (1.1)) and the group manifold (for equation (1.2)) through discrete manifolds. One would expect that such an approximation should be appropriate at large temperatures. The interest of this approximation covers especially the large-N case where Monte Carlo simulations on the original manifolds are difficult.

Let us remind the reader that in the large-N limit,

$$TN = \tilde{T} \qquad \text{(fixed)}, \tag{1.4}$$

the O(N)-model is equivalent to the spherical model (Stanley 1968) and the chiral model (Heller and Neuberger 1982) is equivalent to the quenched model (Eguchi and Kawai 1982).

A discrete approximation to the N-dimensional sphere is the N-dimensional cube which is defined through an N-component vector of unit length

$$\boldsymbol{S}_i^2 = 1 \tag{1.5}$$

and the components are taken over integer numbers. Obviously each component can take the values ± 1. With this constraint, the O(N)-model becomes the discrete cubic model (Aharony 1977). The symmetry of the model is that of the N-dimensional cube W_N (the German word for cube is Würfel). This is the discrete subgroup of O(N) which is obtained taking $N \times N$ orthogonal matrices over integers. The group has $2^N N!$ elements and is isomorphic to the group $Z_2 \wr S_N$ (the wreath product of Z_2 with the permutation group of N objects). Notice that if we had taken higher powers of $\boldsymbol{S}_i \boldsymbol{S}_j$ in equation (1.1), in the discrete approximation all odd (even) powers would be lumped together in the first (second) term in (1.1).

A discrete approximation to the chiral O(N)-model (equation (1.2)) is obtained by taking the W_N instead of the O(N) manifold. This defines the discrete chiral cubic system. The properties of the W_N groups have been intensively studied by Baake (1984). A remarkable property of these groups is that for any N, the N-dimensional vector and the $\frac{1}{2}N(N-1)$-dimensional adjoint irreducible representations of O(N) remain irreducible if one considers the subgroup of W_N. This allows us to give a meaningful approximation to the Hamiltonian (1.2) if we restrict the sum over the irreducible representations to two of them:

$$-H = \sum_{\langle i,j \rangle} (a_V \chi_V(g_i g_j^{-1}) + a_A \chi_A(g_i g_j^{-1})) \qquad (g \in W_N) \tag{1.6}$$

where V and A denote the vector and adjoint representations.

We have not studied the possibility of discretising consistently (for all N!) other coset spaces or group manifolds.

In §§ 2 and 3 we consider in detail the properties of cubic and chiral cubic models for small values of N in various dimensions. We restrict ourselves to the choice $K_2 = 0$ in equation (1.1) and $a_A = 0$ in equation (1.6). The study was done using both mean-field and Monte Carlo analysis combined with finite-size scaling (see Badke *et al* 1985 for the procedure). We find interesting systems whose properties are summarised in § 5. We discover that at a *given dimensionality* for larger values of N the mean-field approximation gets better and better and that one gets first-order phase transitions with an increasing latent heat. This strongly suggests that the large-N limit of the cubic and chiral cubic models is mean field. We make our statement more precise: we conjecture that the large-N

$$T \ln N = \tilde{\tilde{T}} \text{ (fixed)} \tag{1.7}$$

limit of the cubic and chiral cubic models is mean field. This would generalise a similar theorem valid for the N-states Potts model (Mittag and Stephan 1974, Pearce and Griffiths 1980, Cant and Pearce 1983).

We notice that since our systems undergo a first-order phase transition, for a certain value of $\tilde{\tilde{T}}$ say $\tilde{\tilde{T}}_c$, the approximation of the continuous manifold through a discrete manifold becomes useless for large N. Indeed, comparing equations (1.4) and (1.7) one notices that for continuous systems one gets a fake first-order phase transition at

$$\tilde{T}_c = \tilde{\tilde{T}}_c N/\ln N. \tag{1.8}$$

Since the approximation can be applied only for $T > \tilde{T}_c$ its possible domain of applicability is more restricted at large N.

In order to check our conjecture, in § 4 we have considered in $d = 2$ the cubic model given by equation (1.1) with both coupling constants K_1 and K_2. In two dimensions, the system (1.1) is dual to another system with $S_N \wr Z_2$ symmetry (Badke *et al* 1984). We have determined using mean field for both the original and the dual system, the phase diagrams for various N. In this way one gets upper and lower estimates for the critical temperatures. Interestingly enough for large N the phase diagram becomes a line of fixed $\tilde{\tilde{T}}_c$ corresponding to the large-N limit of the $2N$-states Potts model which is mean field.

2. The discrete cubic model

We first write the Hamiltonian (1.1) in a more convenient form (Aharony 1977):

$$-H = NJ \sum_{\langle ij \rangle} [(1-2z) + 2z\delta(\alpha_i - \alpha_j)]\delta(\beta_i - \beta_j) \tag{2.1}$$

where

$$\alpha = 0, 1; \qquad \beta = 0, 1, \ldots, (N-1) \tag{2.2}$$

and

$$K_1 = NJz; \qquad K_2 = NJ(1-z). \tag{2.3}$$

The phase diagram for $N = 3$ and $d = 2$ and $0 \leqslant z \leqslant 1$ has been determined by Badke *et al* (1984). Here we specialise to the point $z = 1$. The model for this special choice of z has been proposed by Kim *et al* (1975), Kim and Levy (1975) and its experimental implications discussed by Kim *et al* (1976).

The system undergoes one phase transition and the values of critical temperatures obtained from mean-field and the Bethe–Peierls–Weiss approximations (Kim and Levy 1975) are given in table 1. In the mean-field approximation one gets a continuous phase transition for $N = 2$ (Ising) and $N = 3$ (with $\alpha = \frac{1}{2}$) and a first-order transition for $N \geqslant 4$ whereas in the BPW approximation the transition is first order for $N \geqslant 3$.

We have performed a Monte Carlo combined with finite-size scaling analysis of the model (see Badke *et al* (1984) for details). For $d = 2$ we find a continuous phase transition for $N = 2$, 3 and 4 and first order for $N \geqslant 5$. For $d = 3$ we get first-order transitions for $N \geqslant 4$ and for $d = 4$ we get first-order transitions for $N \geqslant 3$. The case $d = 4$ and $N = 3$ is marginal in the sense that the latent heat is small and the data are not incompatible with a second-order transition with a large α.

The critical points are given in table 1 and the critical exponents determined from the measurements of the susceptibility (γ/ν) and the specific heat (α/ν) are given in table 2. (We have assumed the hyperscaling relations.)

Table 1. The critical temperature T_c for the cubic model for various N and d. The Monte Carlo calculations were done on lattices of size n^d. The results for $d=3$, $N=2$ are from Sykes *et al* (1972) and for $d=4$, $N=2$ from Gaunt *et al* (1979).

			kT_c/dNJ		
d	N	η	Monte Carlo	BPW	Mean field
2	2	exact (Ising model)	0.5673	0.721	1
	3	7, 10, 14, 20, 30	0.46 ± 0.01	0.497	0.666
	4	5, 7, 10, 14, 17, 20, 22, 30	0.402 ± 0.002	0.394	0.528
	5	7, 10, 14, 20, 30	0.370 ± 0.006	0.335	0.462
3	2		0.7518 ± 0.0001	0.822	1
	3	3, 4, 5, 6, 7, 8, 9, 10, 12	0.55 ± 0.01	0.555	0.666
	4	3, 4, 5, 6, 7, 8, 9, 10	0.458 ± 0.003	0.426	0.528
4	2		0.835 ± 0.002	0.869	1
	3	3, 4, 5, 6	0.577 ± 0.003	0.583	0.666

We first notice that when comparing the Monte Carlo results with those from mean-field and the BPW approximations the data are closed for fixed N and higher d (this is to be expected) and at fixed d for high N. This suggests that at fixed d the large-N limit of the cubic model (here for $z=1$) is mean field. We will come back to this problem in § 4.

We next look at the critical exponents of table 2. For $d=2$ we recover the critical exponents obtained in the previous paper (Badke *et al* 1984). For $d=3$ the results are new. Notice that we have obtained a large value for α. As suggested by Kim *et al* (1976), the model can be used to describe the 5.4K phase transition in HoSb. There the specific heat is fitted with $\alpha = 0.85 \pm 0.1$ on one side of the critical point and $\alpha' = 0.54 \pm 0.01$ on the other side of the critical point. If one takes the first value seriously it is on top of the theoretical prediction.

Table 2. Critical exponents for the cubic model in two and three dimensions.

d	N	η	α/ν	α
2	3	0.32 ± 0.08	0.99 ± 0.06	0.66 ± 0.03
	4	0.33 ± 0.05	1.35 ± 0.05	0.81 ± 0.02
3	3	-0.4 ± 0.2	2.137 ± 0.025	0.832 ± 0.006

3. The chiral cubic model

The models are defined by the Hamiltonian (1.6). We restrict our study to the case $a_A = 0$. If we denote by O an $N \times N$ orthogonal matrix over integers the Hamiltonian can be written:

$$-H = \sum_{\langle ij \rangle} \mathrm{Tr}(O_i O_j^{\mathrm{T}}) \qquad (O_i \in \mathrm{W}_N). \tag{3.1}$$

Table 3. Critical points (T_c), latent heats (ΔE) and discontinuities of the magnetisation (Δm) for the W_N model in the mean-field approximations.

N	KT_C/d	$\Delta E/dN$	Δm
1	2	0	0
2	1.0567	0.4572	0.5791
3	0.8039	0.7557	0.7841
4	0.6844	0.8686	0.8603

For $N=1$ we get the Ising model. There are $2^N N!$ matrices per lattice point. We define an order parameter m:

$$m=\frac{1}{V}\sum_i \det(O_i). \tag{3.2}$$

We have first performed a mean-field calculation. For $N \geqslant 2$ one obtains a first-order phase transition. The critical points, the latent heats ΔE and the discontinuities in the magnetisation m are given in table 3.

Before we present our calculations we notice that for $N=2$ and $N=3$ the group can be written as a semi-direct product of Abelian groups. Now there is a theorem (Drouffe *et al* 1979) that any character function on a group which is a semi-direct product of Abelian groups can be written as a character function on the Abelian groups themselves. This allows us in the case $N=2$ to map the eight-states system (3.1) on an eight-states system with $Z_2 \wr Z_2 \wr Z_2$ symmetry. More precisely this corresponds to the point $x=0$ in equation (2.10) of Badke *et al* (1984) and all we have to do is to copy the result for $d=2$.

Thus for $N=2$, $d=2$ we have a second-order transition with

$$\alpha=0.81\pm0.02; \qquad \eta=0.33\pm0.05.$$

Our Monte Carlo results for $d=2$ are shown in table 4 for $N=2, 3, 4$ and in table 5 for $d=3$ and $N=2, 3$. We notice the same pattern as in the previous section. At

Table 4. Critical points, latent heats and magnetisations for the W_N model in $d=2$ (Monte Carlo results).

N	$KT_C/2$	$\Delta E/2N$	Δm
1	1.135	0	0
2	0.804 ± 0.004	0	0
3	0.67 ± 0.01	0.52 ± 0.01	0.86 ± 0.01
4	0.61 ± 0.02	0.63 ± 0.02	0.92 ± 0.02

Table 5. Critical points, latent heats and magnetisations for the W_N model in $d=3$ (Monte Carlo results).

N	$KT_C/3$	$\Delta E/3N$	Δm
1	1.5036 ± 0.0001	0	0
2	0.916 ± 0.005	0.39 ± 0.01	0.68 ± 0.02
3	0.72 ± 0.01	0.67 ± 0.02	0.86 ± 0.01

fixed N the mean-field results approach the Monte Carlo ones at large d and at fixed d, the mean field approaches the correct result for larger N.

The large-N limit of the mean field gives a first-order phase transition at

$$T_c = d/k \ln N. \tag{3.3}$$

We conjecture that in the large-N limit (fixed d) the chiral cubic models have a first-order transition at the same critical temperature.

4. The large-N limit of the cubic model in two dimensions

In order to get a better insight into the large-N limit behaviour of the discrete cubic model given by equation (2.1), we notice that in $d = 2$ this model is dual to a model with $S_N \wr Z_2$ symmetry given by the Hamiltonian

$$-H = 2\tilde{J} \sum_{\langle ij \rangle} [(1-2\tilde{z}) + N\tilde{z}\delta(\beta_i - \beta_j)]\delta(\alpha_i - \alpha_j). \tag{4.1}$$

The correspondence between J, z and $\tilde{J}$ and $\tilde{z}$ is given in equation (4.3) of Badke *et al* (1984).

The point $z = \tilde{z} = \frac{1}{2}$ in both models corresponds to the $2N$-states Potts model where we know the exact critical temperature (Wu 1982):

$$T_c = \frac{dJ}{k}\left(\frac{\ln(2N)}{N} \cdot \frac{\ln[(2N)^{1/2}+1]}{\ln(2N)^{1/2}}\right)^{-1}. \tag{4.2}$$

In the large-N limit one finds:

$$T_c \approx \frac{dJ}{k}\left(\frac{\ln(2N)}{N}\right)^{-1}\left(1 - \frac{2}{(2N)^{1/2}\ln(2N)}\right). \tag{4.3}$$

We have performed mean-field calculations on both the original model (equation (2.1)) and the dual model (equation (4.1)) for $N = 3, 4, 5$ and 6. The results are shown in figure 1.

The normalisation was chosen in such a way that (following equation (4.2)) the exact value for $z = \frac{1}{2}$ is at 1.0.

The two mean-field calculations give upper and lower bounds for the true critical temperatures. Actually we have noticed in the case $N = 3$ where the correct critical temperatures are known from Monte Carlo calculation (Badke *et al* 1984) that taking the average of the two phase diagrams obtained from the mean field gives results very close to the correct ones.

From figure 1 we learn that for large N the phase diagram reduces to a first-order straight line parallel to the z axis corresponding to the critical temperature of the $2N$-states Potts model:

$$T_c = \frac{dJ}{k}\left(\frac{\ln(2N)}{N}\right)^{-1} \tag{4.4}$$

The second-order Ising line is squeezed at $z = 0$.

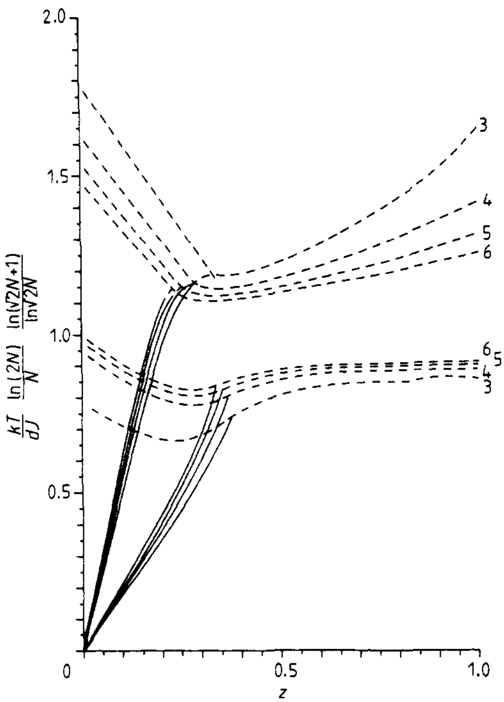

Figure 1. The phase diagrams for the cubic model (upper curves) and its dual (lower curves) for various values of N. The second-order lines are Ising. The broken lines denotes a first-order phase transition; the full lines denote a second-order phase transition.

For completeness we give the large-N behaviour of mean field for the discrete cubic model:

(a) $z<\frac{1}{4}$

$$\frac{kT_c}{Jd}\frac{\ln(2N)}{N}=1+\frac{1}{(2N)^{4z}\ln(2N)}+\ldots \tag{4.5a}$$

$$\Delta E/dN=1-[4z/(4N)^{4z}]+\ldots$$

(b) $z>\frac{1}{4}$

$$\frac{kT_c}{Jd}\cdot\frac{\ln(2N)}{N}=1-\frac{(1-z)}{N}+\frac{1}{2N\ln(2N)}+\ldots \tag{4.5b}$$

$$\Delta E/dN=1+[(2-z)/N]+\ldots.$$

Notice the different behaviour for z larger and smaller than $\frac{1}{4}$ which can also be observed in figure 1.

5. Conclusions

We think that we have clarified the properties of the phase transitions for both the discrete cubic and the chiral cubic models for various N. The results are given in tables 1-5. One striking new result is the critical exponents for the $N=3$ cubic system in $d = 3$ dimension. The value of α is in agreement with the experimental data (Taub *et al* 1974).

The large-N behaviour of both systems (fixed d) suggests that, in this limit, the mean field is exact. We can speculate that the large-N behaviour of any discrete system is mean field unless the large-N limit of the symmetry group is a continuous group. In the large-N limit, the S_N groups (N-states Potts model), the $Z_2 \wr S_N$ groups (cubic models) and the W_N groups (chiral cubic models) are not continuous groups. This is not the case for the vector Potts model (Elitzur *et al* 1979, Cardy 1980) where the large-N limit of Z_N is a continuous group U(1) and the limit in this case is not mean field.

References

Aharony A 1977 *J. Phys. A: Math. Gen.* **10** 389
Baake M 1984 *J. Math. Phys.* **25** 3171
Badke R, Reinicke P and Rittenberg V 1985 *J. Phys. A: Math. Gen.* **18** 73
Cant A and Pearce P A 1983 *Commun. Math. Phys.* **90** 373
Cardy J L 1980 *J. Phys. A: Math. Gen.* **13** 1507
Drouffe J M, Itzykson C and Zuber J B 1979 *Nucl. Phys.* B **147** 132
Eguchi T and Kawai H 1982 *Phys. Rev. Lett.* **48** 1063
Elitzur S, Pearson R B and Shigemitsu J 1979 *Phys. Rev.* D **19** 3698
Gaunt D S, Sykes M F and McKenzie S 1979 *J. Phys. A: Math. Gen.* **12** 871
Heller U M and Neuberger H 1982 *Phys. Rev. Lett.* **49** 621
Kim D and Levy P M 1975 *Phys. Rev.* B **12** 5105
Kim D, Levy P M and Sudano J J 1976 *Phys. Rev.* B **13** 2054
Kim D, Levy P M and Uffer L F 1975 *Phys. Rev.* B **12** 989
Mittag L and Stephen M J 1974 *J. Phys. A: Math. Gen.* **7** L109
Pearce P A and Griffiths R B 1980 *J. Phys. A: Math. Gen.* **13** 2143
Stanley H E 1968 *Phys. Rev.* **176** 718
Sykes M F, Gaunt D S, Roberts P D and Wyles J A 1972 *J. Phys. A: Math. Gen.* **5** 640
Taub H, Williamson S J, Reed W A and Hsu F S L 1974 *Solid State Commun.* **15** 185
Wu F Y 1982 *Rev. Mod. Phys.* **54** 235